

THE FERRIS WHEEL AND JUSTIFICATIONS OF CURVATURE

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This report discusses the results of semi-structured clinical interviews with ten prospective secondary mathematics teachers who were provided with dynamic images of Ferris wheels. We asked the students to graph the relationship between the distance a rider traveled around the Ferris wheel and the height of the rider from the ground. We focus on the different quantitative and non-quantitative ways of thinking in which students engaged to justify the curvature of their drawn graphs. We also discuss how these ways of thinking relate to reasoning covariationally about directional change and amounts of change.

Keywords: Modeling, Cognition, Problem Solving

Introduction

Authors of the Common Core State Standards for Mathematics (CCSSM) (National Governors Association Center for Best Practices, 2010) recommended that educators provide students with repeated and sustained opportunities to model situations via constructing and comparing relationships including those involving constant and changing rates of change. Researchers have also shown that quantitative reasoning (i.e., the analysis of a situation into a quantitative structure (Thompson, 2011)) and covariational reasoning (i.e., students conceiving situations as composed of measurable attributes that vary in tandem (Carlson, M. P., Jacobs, S., Coe, E., Larsen, S., & Hsu, E., 2002)) are critical for numerous K-16 topics (Ellis, 2007; Johnson, 2015; Moore & Carlson, 2012). These researchers have argued for more research to understand how students reason when constructing and comparing relations between quantities.

Given that modeling and reasoning about rates of change are important components of mathematical reasoning, this study aims to answer researchers' calls to explore and to gain a better understanding of students' reasoning about quantities given modeling situations. Namely, we draw on the framework proposed by Carlson et al. (2002) with a particular focus on students' mental actions of directional change and amounts of change. We analyze students' reasoning against the backdrop of these covariation mental actions in the context of their justifying the curvature of graphs they constructed based on a given dynamic situation. Prior research on univariational reasoning and shape thinking also inform our analysis of students' reasoning.

Literature Review

Carlson et al. (2002, pp. 357-358) described a framework explicating students' covariational reasoning that includes five mental actions. This study focuses specifically on Mental Action 2 (MA2) and Mental Action 3 (MA3) of Carlson and colleagues' framework. MA2 is "coordinating the direction of change of one variable with changes in the other variable" (Carlson et al., 2002, p.357). Behaviors indicative of MA2 include "constructing an increasing straight line" and verbalizing "the direction of change of the output *while* [emphasis added] considering changes in the input" (Carlson et al., 2002, p. 357). MA3 involves considering (and possibly comparing) amounts of change in one quantity for amounts of change in another quantity. Some researchers (Castillo-Garsow, Johnson, & Moore, 2013; Johnson, 2012) have clarified the ways in which students reason about amounts of change. For example, Johnson (2012) provided insights into particular ways students reason about the intensity of varying quantities by "systematically varying one quantity and simultaneously attending to variation in the intensity of change in a quantity indicating a relationship between covarying quantities" (p. 327). Castillo-Garsow et al. (2013) further distinguished students imagining

the intensity of change in terms of successive chunks or in terms of a smooth, dynamic image of covariation.

Moore and Thompson (2015) clarified students' covariational reasoning in the particular context of graphs. They termed students' *emergent shape thinking* to be a student conceptualizing a graph (in ways potentially compatible with those described in the previous paragraph) as an in-progress trace of two united varying quantities. A student thinking emergently justifies the curvature of a drawn graph in terms of covarying quantities or magnitudes represented along the relevant axes. Highlighting that educators cannot take students' covariational reasoning as a given, Moore and Thompson (2015) identified other ways of thinking that we show students might use when justifying the curvature of a graph. The authors termed students' *static shape thinking* – “operating on a graph as an object in and of itself” (p. 784). Students' static shape thinking includes both iconic translations and thematic associations. Iconic translations involve associations between the visual features of the situation and graph (Monk, 1992), and thematic associations are between the graph and the event phenomena (Thompson, 2015). Lastly, one example of a quantitative justification is univariational reasoning, in which a student considers how one quantity's value changes without any systematic coordination with another quantity's value (Leinhardt, Zaslavsky, & Stein, 1990).

Theoretical Framework

We approach knowledge as actively constructed in ways idiosyncratic to the knower; an individual's knowledge is fundamentally unknowable to another (von Glasersfeld, 1995). When referring to a quantity, we do not imply that a quantity exists independent of a person conceiving it. Quantities are personally constructed measurable attributes (Steffe, 1991; Thompson, 2011). Similarly, relationships between quantities are constructed by the individual; the relationship hinges of the individual's understanding of each quantity and their image of the situation, which may not be compatible with a researcher's intentions (Moore & Carlson, 2012). Although an individual's knowledge is taken to be fundamentally unknowable to another person, it is possible for a person to make inferences about another individual's thinking. These inferences are based on the person's interpretations of the words and actions of the individual. These inferences are the basis of the person's constructed model of an individual's mathematics, referred to as *the mathematics of students* (c.f., *students' mathematics*) (Steffe & Thompson, 2000).

Methodology

This study focuses on student work from one problem during semi-structured clinical interviews (Clement, 2000; Goldin, 2000). We conducted the study with ten students (eight female, two male) who were enrolled in a secondary mathematics teacher education program at a large university in the southeast United States. The problem consisted of two tasks given during the first of three interviews with each student. At the time, the students were either enrolled in or had completed their first content course in the secondary mathematics education program, and all the students had completed a calculus sequence and at least two additional mathematics courses (e.g., linear algebra, differential equations, etc.) with at least C letter grades. Some of the students had completed additional courses in both mathematics and mathematics education, but these courses did not focus on ideas directly related to the interview tasks. Two members of the research team were present at each interview, and each interview was videotaped and digitized for analysis. We used an open (generative) and axial (convergent) approach based on grounded theory (Corbin & Strauss, 2008) and in combination with conceptual analysis (Thompson, 2008; von Glasersfeld, 1995) to analyze the results. That is, we used an iterative approach to construct viable models of the subjects' thinking and their ways of reasoning.

Task Design – The Ferris Wheel

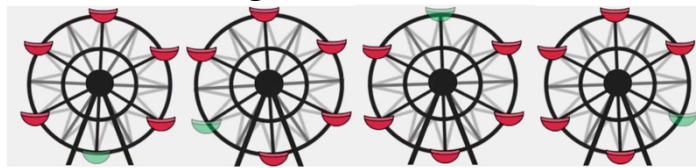


Figure 1. Animation Snapshots of the Ferris Wheel Task I and Task II.

This study focuses on the Ferris wheel problem, which is split into two tasks. First, students view an animation of a Ferris wheel rotating clockwise (Desmos, 2014) continuously; the Ferris wheel has a green cart that begins at the bottom to represent the location of the rider (Figure 1). He or she then responds to Task I: “Graph the relationship between the total distance the rider has traveled *around* the Ferris wheel and the rider’s distance from the ground.” Once completed, the student views an animation of a Ferris wheel that moves at a slower speed and pauses three times at the locations seen in the latter three snapshots (Figure 1). We then provide Task II: “Watch the video, which dictates a second rider’s trip around the Ferris wheel. Is the previous graph relevant? Explain.” The goal of this problem is to (1) provide a student with two situations with perceivable differences, yet the same underlying relationship (ignoring explicit parameterizations) between two distances, and (2) afford MA3; for successive equal amounts of distance traveled around the Ferris wheel for the first quarter of a trip, the rider’s height above the ground increases and the amount by which the height increases is also increasing.

Results

We organize students’ justifications for the curvature of their graph into two categories: non-quantitative justifications and quantitative justifications. A non-quantitative justification involves reasoning in which some non-quantitative aspect of the given situation implies or necessitates a (possibly perceptual) feature of the drawn graph. A quantitative justification includes some consideration of the quantities and changes in at least one of the conceived quantities. We are not implying that a student providing non-quantitative justifications did not reason covariationally or quantitatively, only that some aspect of their justification was non-quantitative. For instance, students commonly justified their graphs via directional change (MA2), but provided non-quantitative justifications when prompted to explain the curvature of their drawn graphs further.

Non-Quantitative Justification of Curvature

The first example of a non-quantitative justification for the curvature of the graph is consistent with students’ static shape thinking and iconic translation. In other words, the student relies on shapes in the situation to justify his or her drawn graph. In this task, students who exhibited static shape thinking justified the curvature of the graph based on the shape of the wheel (i.e., circular wheel implies circular graph). To illustrate, Rory created her graph (Task I) drawing a curved shape, but only described directional change (MA2) when justifying her graph:

R: It starts total distance and then it gets higher about half way, so say this [marks point on the top center of her graph] is our total distance...It peaks about half way, so, however many seconds, um, and then it just goes up, I would say, uh, at a quadratic rate [plots points on the left side of her graph and connects them with a curved line, completes right side of her graph]...it goes up and down in sort of a parabola shape [Figure 2a].

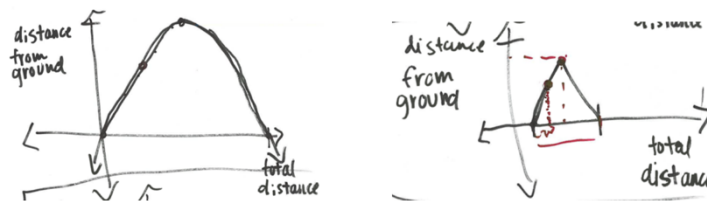


Figure 2. (a, left) Rory's First Graph and (b, right) Second Graph.

When asked to explain the “quadratic rate” further, Rory explained her “parabola shape”:

R: ...As we go up higher and higher, the total distance increases...but when we come down about that half-way point where the [cart] is, um, exactly 90 degrees in the opposite direction, so where it just passed [at the top of the Ferris wheel], um, that's where it peaks [pointing to the top of her graph] and that's where it's the highest, but it's only about half the total distance that it's gone. So, it could also reflect, I guess, an absolute value graph [draws Figure 2b excluding axes and labels], but a downward facing upside down absolute value graph...I feel like because it's more circular in nature, and not like necessarily linear, um, to where it, I guess it would be like a hexagon or something with lines connecting, I would feel more parabola shaped just because it's curved, rather than an absolute value. But I could see how it would be reflecting an absolute value graph as well, so. But, I would still go with my parabola or quadratic shape as I was saying...

I: ...the parabola shape because you said it's, what's curved? Something's curved?

R: The, the Ferris wheel in and of itself is a circle.

We infer that Rory understands that both a linear (i.e., “absolute value”) and a curved graph (i.e. “quadratic”) capture the appropriate directional relationship (MA2). However, her explanation for why she chose to draw a curved graph is based on the shape of the Ferris wheel. Thus, she engaged in both covariational reasoning and an iconic translation, with the iconic translation implying that the graph be “parabola” and not “linear” in shape.

A second example of a non-quantitative justification is thematic shape thinking. Students' thinking of graphs thematically entail their making associates between some perceived phenomenon of a situation and features of a graph (Thompson, 2015); in such a case, relationships between quantities are *implications* of assimilating some perceived phenomenon (Moore & Thompson, 2015). In the case of Task II, a student thinking thematically could involve his or her associating pauses that occur in a trip with some perceptual feature in the graph (e.g., horizontal lines, bolded points, or breaks in the graph) that “shows” the pauses.

Moreover, students might note the constant rotational speed of the Ferris wheel and make associations with the graph on the basis of constant speed (i.e., students conclude that constant speed unquestionably implies a graph composed of linear segments, and varying speeds unquestionably imply a graph composed of non-linear curves). For example, after Rory explicitly stated “relationships don't necessarily mean anything to do with the, the actual physical being being circular” after the previous excerpt, she abandoned her graph (Figure 2a) and watched the Ferris wheel again. Then, she made the following observation about the Ferris wheel:

R: It looks more at a constant rate now that I look at it. So it would probably be more of this absolute value shape... [completes Figure 2b]... And I don't, again, if it speed up, sped up like at some point to where the higher it went, it would represent more of the quadratic shape [pointing to Figure 2a graph]. So I would go with this graph [Figure 2b graph].

Rory perceived the Ferris wheel rotating at a constant speed and concluded that her graph should now have an “absolute-value shape” (i.e., linear). Notice also that her language in describing her

graph is still based on shapes (i.e., the absolute value *shape* and the quadratic *shape*). Because Rory's justification relied on a perceived phenomenon (i.e., constant speed) implying a shift in a perceptual feature in her (distance-distance) graph (i.e., *quadratic* to a *absolute-value shape*), we infer her actions to indicate thematic shape thinking.

Quantitative Justification of Curvature

Univariational Reasoning. When a student engages in univariational reasoning, he or she alludes to and possibly compares changes in some quantity to make a claim about curvature. This comparison involves looking at different regions of the graph, although not systematically with respect to changes in another quantity, as Johnson (2012) described. For instance, in the Ferris wheel problem, students imagined that the rider's distance from the ground appears to “stay longer” at the top of the Ferris wheel than in the middle, concluding that there should be a difference in how these two regions appear on their graphs. Although their justifications considered the magnitude of the height and how this magnitude changed in some way in certain regions of the graph, the students did not explicitly or systematically hold in mind variations of the total distance traveled around the Ferris wheel; the students often made these comparisons for implicit total distance intervals of unequal sizes or interval sizes that are too large (in the sense that the relationship between the two quantities changes within the chunk under consideration).

For instance, consider Louise who initially drew her graph (Figure 3a without numerical values) considering directional changes in both quantities (i.e., “distance around [the Ferris wheel] is always increasing” and “distance from the ground grows and then stops and then goes back down” for one rotation around the wheel). When we asked her to justify the curvature of her graph, she considered a graph with a “peak” similar to Figure 2b. At one moment, she mentioned that a curved graph “looks better” but that maybe she is “just thinking that ‘cause it’s a curved circle [referring to the shape of the Ferris wheel].” Unsatisfied with this conclusion, she considered how the height changes in a way that we consider indicative of univariational reasoning. The following transcript includes her final justification for the curvature of her graph:

- L: It's not like curved curved, but it's sorta curved. It's a small curve. Just because, it's. Because the distance around if you look at the top [of the wheel]...this whole area is kind of the peak [points to range at the top of the wheel], a little bit. Like it kind of like travels at the top, just a little bit. Um, so I think it kinda travels at the top here a little bit.
- I: What about like through here [pointing to the line segment between 0 and 5, Figure 3a]?
- L: Well, I think, 'cause it's continually growing. I mean like, coming from the ground it's not really staying anywhere. Like, it's not stopping there [pointing at location along left side of the wheel] and being like, “Oh, I'm gonna chill here”. You know. It's constantly going, but at the top it kind of has to like go around the curve, so it's at the peak for a little longer than on here [motioning along the right side of the Ferris wheel]. And the same thing down here [motioning along the bottom of the Ferris wheel]. That's kinda why it's curved down here as well [pointing to the bottom of Figure 3b]. Because it, you know, stays at the bottom for just a, just like a tad bit longer than it would on the sides.

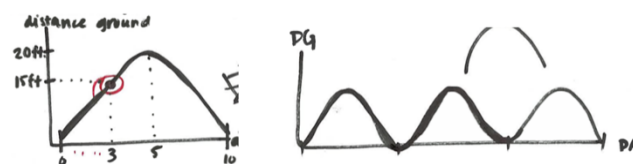


Figure 3. (a) Louise's Initial Graph and (b) Graph for Three Revolutions.

Louise split the Ferris wheel into four sections to justify the curvature of her graph: the top and bottom sections and the two sides. Within each section, she imagined the height with respect to the experience of the ride to draw conclusions about the intensity of height changes. Because the height seemed to “stay” near the top and bottom sections “longer”, she concluded that these sections of her graph are curved. For the sides of the Ferris wheel, she discussed the height as “constantly” changing; she represented those sections on her graph with (nearly) linear segments. Overall, her description included the intensity of one quantity’s changes in one of her four sections (of unspecified size) without referencing how the second quantity (i.e., distance traveled around the wheel) related to these changes in height. This reasoning contrasts with the covariational reasoning Johnson (2012) described, which involves reasoning about the intensity of a quantity because Louise is not systemically comparing the intensity of height changes to another quantity (e.g., intensity of changes in distance traveled around the Ferris wheel).

Conflated Amounts of Change. When a student conflates amounts of change, he or she compares the changes in one quantity with respect to another quantity by considering how equal successive changes in one quantity relate to variations in the other quantity. However, he or she may be unsure how to imagine or represent the given quantities and their changes in the situation or the graph. The former leads to considering changes of a quantity in the situation that we, as observers, consider different than requested. For the Ferris wheel problem, a student might depict equal changes of “total distance” along the horizontal diameter of the wheel instead of equal changes in “total distance” around the wheel. The latter leads to students (possibly unknowingly) conflating which axes they use to represent particular quantities.

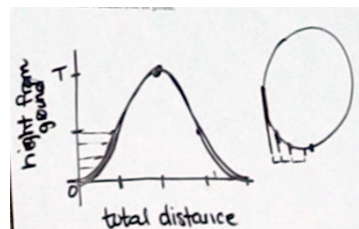


Figure 4. Kim's Graph and Diagram.

Kim began Task I by plotting points for each quarter turn and then each remaining one-eighth turn of the Ferris wheel. She connected the points using curved lines (Figure 4). She indicated reasoning directionally (MA2) when she described “he comes back down” and “he’s back at 0” while tracing along the arc of the right side of the Ferris wheel.

When asked to justify connecting her plotted points, she immediately began drawing a diagram of the situation in order to relate the situation to her graph (Figure 4). In her explanation, she made two confluents. First, turning to her diagram, Kim marked off equal changes *along the ground* (as indicated by the horizontal segments in her diagram in Figure 4) and compared changes in height. This reasoning led her to conclude that her graph needed to show “height was increasing at an increasing rate”. Second, when she turned to the corresponding section in her graph, she marked equal changes along the vertical axis (i.e., equal changes in the axes labeled *height from the ground*) and appropriately concluded that she “drew it like it’s increasing at a decreasing rate, though.” Thus, the equal changes she was comparing in both situations did not represent the same quantity (from our perspective). However, in both her diagram and her graph, she made appropriate conclusions for the relationships between the quantities she was comparing (e.g., with respect to the drawn graph, abscissa values do increase at a decreasing rate as ordinate values increase from 0). Although Kim was able to reason with the given quantities in order to plot points on her graph and reason using

MA2, she conflated the quantities when attempting to reason about amounts of change to justify the curvature of her graph.

Amounts of Change. We argue that a student reasoning emergently about the curvature of their graph with respect to the situation they are modeling occurs when he or she is able to exhibit MA3 on the given situation (or a student's diagram of the situation) and then create/modify/justify his or her graph in terms of such a relationship. For example, April justified her graph as she drew it. She marked equal changes of “arc length or total distance around” the Ferris wheel, identified the change in height for each interval using the situation, made conclusions about rates of change for given intervals around the wheel, and finally indicated how these changes are represented on her graph (Figure 5).

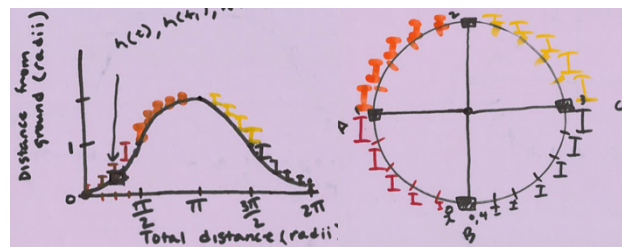


Figure 5. April's Justification of Curvature Using Amounts of Change.

Discussion

This paper focuses on types of reasoning students engage in when modeling a dynamic situation. The results corroborate several of Carlson et al.'s (2002) and Saldanha and Thompson's (1998) claims. Overall, many students (8/10) reasoned directionally (MA2) from the researcher's perspective, but they often constructed graphs without considering amounts of change *prior* to their drawing a graph; several students (8/10) had difficulty (quantitatively) justifying the curvature of their drawn graphs. In our study, we identified specific ways of thinking students exhibited when attempting to justify the curvature of their graphs. Students who reasoned non-quantitatively exhibited forms of static shape thinking when asked to justify the curvature (or lack thereof) of their graphs. We also noticed students who took a quantitative approach. However, before students could reason effectively about amounts of change (MA3), they needed to construct not only the change in one quantity, but also the change in that quantity with respect to a change in another given quantity. Students who were only reasoning about the change in one quantity without systematically considering changes in another quantity reasoned univariationally. Furthermore, students who realized that they needed to compare the change in one given quantity with respect to a change in another given quantity were not always sure how to represent these quantities on either their graph or the situation, often conflating quantities from our perspective. Piaget (1974) described that the path to consciousness is a dynamic, active system, which is consistent with the idea that quantities are not intrinsic attributes of graphs and situations. Furthermore, he claimed, “knowledge does not proceed from the subject or from the object, but from interaction between the two” (p. 335). Providing students with tasks that afford opportunities to reason quantitatively and covariationally encourages students constructing situations constituted by quantities, and perhaps helps to promote the construction of quantitative MA3 reasoning.

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